

Boson Realization of the $su(3)$ -Algebra. I

— *Schwinger Representation for the Lipkin Model* —

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Abstract

Following a general form for the Schwinger boson representation of the $su(M + 1)$ Lipkin model presented in the previous paper, three types of the orthogonal sets characterizing the $su(3)$ -algebra are proposed. In these three, third is presented in Appendix. The intrinsic state is specified by two quantum numbers and two excited state generating operators play a central role.

§1. Introduction

It is well known that the Lipkin model¹⁾ has played a crucial role in microscopic studies of collective motion observed in nuclei. The original Lipkin model is based on the $su(2)$ -algebra and, with the aid of this model, we can understand schematically collective motion based on superposition of particle-hole pair excitations. However, in order to serve the studies of coupling of collective and non-collective degree of freedom, which is one of the most fundamental problems in nuclear theory, the $su(2)$ -Lipkin model must be generalized and the simplest generalization may be the $su(3)$ -algebraic model.²⁾ On the other hand, this model has been also served as the schematic understanding of finite temperature effects.³⁾ In this case, the use of the free vacuum is helpless. The above situations are mentioned in detail in Ref.4), which is, hereafter, referred to as (A).

A main aim of this paper, (I), is to formulate the Schwinger boson representation for the Lipkin model, which is presented in (A), for the case of the $su(3)$ -algebra in complete form. In (A), only the surface is sketched. The $su(3)$ -algebra consists of eight generators, in which some generators determine the intrinsic state and some of others play a role of the excited state generating operators. With the use of these operators, we can determine the orthogonal set of the $su(3)$ -algebra completely. Depending on the properties of the excited state generating operators, three forms are presented. In §4, §5 and Appendix, the details are discussed. As was already treated in (A), the present $su(3)$ -algebra is described in terms of six kinds of boson operators : two sets of vector types. But, we know that the orthogonal set of the $su(3)$ -algebra is specified by five quantum numbers, and the remaining one is used for the $su(1,1)$ -algebra.

In the next section, the $su(3)$ -algebra in the Schwinger representation is recapitulated in terms of the form presented in (A). Section 3 is devoted to constructing the intrinsic state specified by two quantum numbers. In §4, the first type of the orthogonal set is given by regarding the two excited state generating operators as tensor operators with rank $1/2$ (spinor). The main part of §5 is to construct the second type in terms of successive operations of the operators related to the excited state generating operators on the intrinsic state. In §5, some characteristic points obtained in §§3 and 4 are discussed. In Appendix, the third form is given in the form of successive operations of the excited state generating operators together with certain projection operators.

§2. The $su(3)$ -algebra in the form suitable for the Lipkin model and its Schwinger representation

As was recapitulated in (A), the $su(3)$ -algebra is composed of eight generators, which were, in (A), denoted as $(\hat{S}_2^1, \hat{S}_1^2, \hat{S}_1^1, \hat{S}_2^2, \hat{S}^1, \hat{S}^2, \hat{S}_1, \hat{S}_2)$. In this paper, we formulate the $su(3)$ -algebra in a form, which may be suitable for the boson realization of the Lipkin model. Therefore, the notations different from the above are adopted:

$$\hat{I}_+ = \hat{S}_2^1, \quad \hat{I}_- = \hat{S}_1^2, \quad \hat{I}_0 = (1/2)(\hat{S}_2^2 - \hat{S}_1^1), \quad (2.1a)$$

$$\hat{M}_0 = (1/2)(\hat{S}_2^2 + \hat{S}_1^1), \quad (2.1b)$$

$$\hat{D}_-^* = \hat{S}^1, \quad \hat{D}_+^* = \hat{S}^2, \quad \hat{D}_- = \hat{S}_1, \quad \hat{D}_+ = \hat{S}_2. \quad (2.1c)$$

The above operators obey the following commutation relations:

$$[\hat{I}_+, \hat{I}_-] = 2\hat{I}_0, \quad [\hat{I}_0, \hat{I}_\pm] = \pm\hat{I}_\pm, \quad (2.2a)$$

$$[\hat{I}_{\pm,0}, \hat{M}_0] = 0, \quad (2.2b)$$

$$[\hat{I}_\pm, \hat{D}_\pm^*] = 0, \quad [\hat{I}_\pm, \hat{D}_\mp^*] = \hat{D}_\pm^*, \quad (2.2c)$$

$$[\hat{I}_0, \hat{D}_\pm^*] = \pm(1/2)\hat{D}_\pm^*, \quad (2.2d)$$

$$[\hat{M}_0, \hat{D}_\pm^*] = (3/2)\hat{D}_\pm^*, \quad (2.2d)$$

$$[\hat{D}_+^*, \hat{D}_-^*] = 0, \quad [\hat{D}_\pm^*, \hat{D}_\pm] = \hat{M}_0 \pm \hat{I}_0, \quad [\hat{D}_\pm^*, \hat{D}_\mp] = \hat{I}_\pm. \quad (2.2e)$$

We can see in the relation (2.2a) that the set $(\hat{I}_{\pm,0})$ obeys the $su(2)$ -algebra, and further, the set (\hat{D}_\pm^*) forms a spinor, i.e., a spherical tensor with rank 1/2 with respect to $(\hat{I}_{\pm,0})$, in which + and - correspond to the z -components +1/2 and -1/2, respectively. We can see this fact in the relation (2.2c). The Casimir operator $\hat{I}_{su(3)}$ can be expressed in the form

$$\begin{aligned} \hat{I}_{su(3)} = & \hat{S}^1\hat{S}_1 + \hat{S}_1\hat{S}^1 + \hat{S}^2\hat{S}_2 + \hat{S}_2\hat{S}^2 + \hat{S}_2^1\hat{S}_1^2 + \hat{S}_1^1\hat{S}_2^2 \\ & + (2/3) \left[(\hat{S}_1^1)^2 - \hat{S}_1^1\hat{S}_2^2 + (\hat{S}_2^2)^2 \right]. \end{aligned} \quad (2.3)$$

The form (2.3) can be reexpressed as

$$\hat{I}_{su(3)} = 2 \left[\hat{\mathbf{D}}^* \hat{\mathbf{D}} + \hat{\mathbf{D}} \hat{\mathbf{D}}^* + (1/3)\hat{M}_0^2 + \hat{\mathbf{I}}^2 \right], \quad (2.4)$$

$$\begin{aligned} \hat{\mathbf{D}}^* \hat{\mathbf{D}} = & (1/2)(\hat{D}_+^* \hat{D}_+ + \hat{D}_-^* \hat{D}_-), \\ \hat{\mathbf{D}} \hat{\mathbf{D}}^* = & (1/2)(\hat{D}_+ \hat{D}_+^* + \hat{D}_- \hat{D}_-^*), \end{aligned} \quad (2.5a)$$

$$\hat{\mathbf{I}}^2 = \hat{I}_0^2 + (1/2)(\hat{I}_+ \hat{I}_- + \hat{I}_- \hat{I}_+). \quad (2.5b)$$

Associating the above $su(3)$ -algebra, the $su(1,1)$ -algebra also plays a central role in the present form. It is composed of three generators which were, in (A), denoted as $(\hat{T}^1, \hat{T}_1, \hat{T}_1^1)$. In this paper, we will use the notation $(\tilde{T}_{\pm,0})$ which is defined as

$$\tilde{T}_+ = \hat{T}^1, \quad \tilde{T}_- = \hat{T}_1, \quad \tilde{T}_0 = (1/2)\hat{T}_1^1. \quad (2.6)$$

The commutation relation and the Casimir operator are given as follows:

$$[\tilde{T}_+, \tilde{T}_-] = -2\tilde{T}_0, \quad [\tilde{T}_0, \tilde{T}_{\pm}] = \pm\tilde{T}_{\pm}, \quad (2.7)$$

$$\hat{I}_{su(1,1)} = 2\tilde{\mathbf{T}}^2, \quad (2.8)$$

$$\tilde{\mathbf{T}}^2 = \tilde{T}_0^2 - (1/2)(\tilde{T}_+\tilde{T}_- + \tilde{T}_-\tilde{T}_+). \quad (2.9)$$

A possible boson realization of the above two algebras is obtained in the framework of six kinds of boson operators (\hat{a}, \hat{a}^*) , (\hat{b}, \hat{b}^*) , $(\hat{a}_{\pm}, \hat{a}_{\pm}^*)$ and $(\hat{b}_{\pm}, \hat{b}_{\pm}^*)$. The detail can be found in (A). By changing the notations adopted in (A) to those suitable for the present form such as $\hat{a}^1 \rightarrow \hat{a}$, $\hat{a}_1 \rightarrow \hat{a}_-$, $\hat{a}_2 \rightarrow \hat{a}_+$, $\hat{b} \rightarrow \hat{b}$, $\hat{b}_2^1 \rightarrow \hat{b}_-$, $\hat{b}_1^1 \rightarrow \hat{b}_+$, we have the following expressions:

$$\begin{aligned} \hat{I}_+ &= \hat{a}_+^* \hat{a}_- - \hat{b}_+^* \hat{b}_-, & \hat{I}_- &= \hat{a}_-^* \hat{a}_+ - \hat{b}_-^* \hat{b}_+, \\ \hat{I}_0 &= (1/2) \left[(\hat{a}_+^* \hat{a}_+ - \hat{a}_-^* \hat{a}_-) + (\hat{b}_+^* \hat{b}_+ - \hat{b}_-^* \hat{b}_-) \right], \end{aligned} \quad (2.10a)$$

$$\hat{M}_0 = (\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) + (1/2) \left[(\hat{a}_+^* \hat{a}_+ + \hat{a}_-^* \hat{a}_-) - (\hat{b}_+^* \hat{b}_+ + \hat{b}_-^* \hat{b}_-) \right], \quad (2.10b)$$

$$\begin{aligned} \hat{D}_-^* &= \hat{a}_-^* \hat{b} + \hat{a}^* \hat{b}_+, & \hat{D}_+^* &= \hat{a}_+^* \hat{b} + \hat{a}^* \hat{b}_-, \\ \hat{D}_- &= \hat{b}^* \hat{a}_- + \hat{b}_+^* \hat{a}, & \hat{D}_+ &= \hat{b}^* \hat{a}_+ + \hat{b}_-^* \hat{a}, \end{aligned} \quad (2.10c)$$

$$\begin{aligned} \tilde{T}_+ &= \hat{a}^* \hat{b}^* - \hat{a}_+^* \hat{b}_-^* - \hat{a}_-^* \hat{b}_+^*, & \tilde{T}_- &= \hat{b} \hat{a} - \hat{b}_- \hat{a}_+ - \hat{b}_+ \hat{a}_-, \\ \tilde{T}_0 &= (1/2) \left[(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) + (\hat{a}_+^* \hat{a}_+ + \hat{b}_+^* \hat{b}_+) + (\hat{a}_-^* \hat{a}_- + \hat{b}_-^* \hat{b}_-) + 3 \right]. \end{aligned} \quad (2.11)$$

Calculation of the commutation relations for $(\hat{I}_{\pm,0})$ defined in the relation (2.10a) tells us that the bosons \hat{a}^* and \hat{b}^* are scalars (rank = 0) and $(\hat{a}_-^*, \hat{a}_+^*)$ and $(\hat{b}_-^*, \hat{b}_+^*)$ are spinors (rank = 1/2). Further, it may be important to see that the representations (2.10) and (2.11) give us the relation

$$[\text{any of the } su(3)\text{-generators}, \text{ any of the } su(1,1)\text{-generators}] = 0. \quad (2.12)$$

The Casimir operator (2.4) can be reexpressed in the form

$$\hat{I}_{su(3)} = 2 \left[\tilde{\mathbf{T}}^2 - 3/4 + (1/3)(\tilde{R}_0)^2 \right], \quad (2.13)$$

$$\tilde{R}_0 = (1/2) \left[(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) - (\hat{a}_+^* \hat{a}_+ - \hat{b}_+^* \hat{b}_+) - (\hat{a}_-^* \hat{a}_- - \hat{b}_-^* \hat{b}_-) \right]. \quad (2.14)$$

In (A), \tilde{R}_0 is denoted as \hat{R}_0 . Naturally, we have

$$[\tilde{R}_0 , \text{ any of the } su(3)\text{- and the } su(1,1)\text{-generators}] = 0 . \quad (2.15)$$

The proof of the relation (2.13) is straightforward, but tedious. The above is an outline of the $su(3)$ -algebra and its associating $su(1,1)$ -algebra in the Schwinger representation suitable for the Lipkin model. The system under investigation is composed of six kinds of boson operators.

§3. Construction of the intrinsic state

The present system consists of six kinds of boson operators and we can easily find that there exist totally six commuted operators: \tilde{T}^2 , \tilde{T}_0 , \tilde{R}_0 , \hat{I}^2 , \hat{I}_0 and \hat{M}_0 . Therefore, the eigenstates for the above operators provide a complete orthogonal set for the $su(3)$ -algebra and its associating $su(1,1)$ -algebra. For the task obtaining this orthogonal set, we, first, construct the intrinsic state introduced in (A).

As was discussed in (A), the intrinsic state, which we denote $|m\rangle$, should obey the following condition:

$$\hat{D}_-|m\rangle = \hat{D}_+|m\rangle = \hat{I}_-|m\rangle = 0 , \quad \tilde{T}_-|m\rangle = 0 . \quad (3.1)$$

In the notation adopted in (A), the condition (3.1) can be expressed as $\hat{S}_1|m\rangle = \hat{S}_2|m\rangle = \hat{S}_1^2|m\rangle = 0$ and $\hat{T}_1|m\rangle = 0$. Further, $|m\rangle$ should be an eigenstate for \tilde{T}_0 , \tilde{R}_0 , \hat{I}_0 and \hat{M}_0 . The state $|m\rangle$ which satisfies the above condition is easily obtained:

$$\begin{aligned} |m\rangle &= |T, R\rangle \\ &= \left(\sqrt{(T-3/2+R)!(T-3/2-R)!} \right)^{-1} (\hat{b}_-^*)^{T-3/2+R} (\hat{b}_-^*)^{T-3/2-R} |0\rangle . \end{aligned} \quad (3.2)$$

It satisfies the eigenvalue equations

$$\tilde{T}_0|T, R\rangle = T|T, R\rangle , \quad (3.3a)$$

$$\tilde{R}_0|T, R\rangle = R|T, R\rangle , \quad (3.3b)$$

$$\hat{I}_0|T, R\rangle = -I^0|T, R\rangle , \quad I^0 = (1/2)(T-3/2+R) , \quad (3.3c)$$

$$\hat{M}_0|T, R\rangle = -M^0|T, R\rangle , \quad M^0 = (1/2)(3(T-3/2)-R) . \quad (3.3d)$$

Since the boson numbers are positive-integers, the form of \tilde{T}_0 shown in the relation (2.11) gives us

$$T = 3/2 , 2 , 5/2 , 3 , \dots . \quad (T \geq 3/2) \quad (3.4)$$

Further, the exponents $(T - 3/2 + R)$ and $(T - 3/2 - R)$ appearing in the state (3.2) should be positive-integers and we have

$$R = -(T - 3/2) , \quad -(T - 3/2) + 1 , \quad \dots , \quad (T - 3/2) - 1 , \quad (T - 3/2) .$$

$$(-(T - 3/2) \leq R \leq T - 3/2) \quad (3.5)$$

The relations (3.3c)~(3.5) give us

$$I^0 = 0 , \quad 1/2 , \quad 1 , \quad 3/2 , \quad \dots , \quad (T - 3/2) , \quad (0 \leq I^0 \leq T - 3/2) \quad (3.6)$$

$$M^0 = I^0 , \quad I^0 + 1 , \quad I^0 + 2 , \quad \dots . \quad (M^0 \geq I^0) \quad (3.7)$$

The eigenvalues of $\hat{F}_{su(3)}$ and $\hat{F}_{su(1,1)}$ for the state $|T, R\rangle$ are expressed as

$$\text{the eigenvalue of } \begin{cases} \hat{F}_{su(3)} = 2[(T - 3/2)(T + 1/2) + (1/3)R^2] , \\ \hat{F}_{su(1,1)} = 2T(T - 1) . \end{cases} \quad (3.8)$$

Instead of (T, R) , we can specify the state $|m\rangle$ in terms of (M^0, I^0) . In this specification, we have

$$\text{the eigenvalue of } \begin{cases} \hat{F}_{su(3)} = 2[(1/3)M^0(M^0 + 3) + I^0(I^0 + 1)] , \\ \hat{F}_{su(1,1)} = 2 \cdot (1/2)(M^0 + I^0 + 3) \cdot (1/2)(M^0 + I^0 + 1) . \end{cases} \quad (3.9)$$

The quantum numbers M^0 and I^0 correspond to λ and μ in the Elliott model in the relations $M^0 = \lambda$ and $I^0 = \mu$. The state $|m\rangle$ is also specified by (T, I^0) . In this case, we have

$$M^0 = 2(T - 3/2) - I^0 , \quad R = 2I^0 - (T - 3/2) . \quad (3.10)$$

This specification will play a central role in §4: $|I^0, T\rangle$.

§4. Structure of the excited states

In (A), we mentioned that the excited states constructed on the intrinsic state $|m\rangle$ are composed in terms of the excited state generating operators \hat{S}^1 , \hat{S}^2 and \hat{S}_2^1 , in the present notations \hat{D}_-^* , \hat{D}_+^* and \hat{I}_+ . If including the $su(1,1)$ -algebra, \tilde{T}_+ is added. With the use of these operators, we construct the orthogonal set. First, we note two points. Since $\hat{I}_-|I^0, T\rangle = 0$ and $\hat{I}_0|I^0, T\rangle = -I^0|I^0, T\rangle$, a possible eigenstate of $\hat{\mathbf{I}}^2$ and \hat{I}_0 , which we denote as $|I^0 I_0^0; T\rangle$, is given in the form

$$|I^0 I_0^0; T\rangle = \sqrt{\frac{1}{(2I^0)!} \frac{(I^0 - I_0^0)!}{(I^0 + I_0^0)!}} (\hat{I}_+)^{I^0 + I_0^0} |I^0, T\rangle ,$$

$$I_0^0 = -I^0 , \quad -I^0 + 1 , \quad \dots , \quad I^0 - 1 , \quad I^0 . \quad (-I^0 \leq I_0^0 \leq I^0) \quad (4.1)$$

The above is the first point. Next, we consider the second point. We have already mentioned that (\hat{D}_\pm^*) is a spherical tensor with rank $1/2$, the z -components of which are specified by $\pm 1/2$. Then, a spherical tensor with rank I^1 and z -component I_0^1 , $\hat{D}_{I^1 I_0^1}^*$, is obtained in the form

$$\hat{D}_{I^1 I_0^1}^* = \left(\sqrt{(I^1 + I_0^1)!(I^1 - I_0^1)!} \right)^{-1} (\hat{D}_+^*)^{I^1 + I_0^1} (\hat{D}_-^*)^{I^1 - I_0^1} . \quad (4.2)$$

We can prove the relation

$$\begin{aligned} [\hat{I}_\pm, \hat{D}_{I^1 I_0^1}^*] &= \sqrt{(I^1 \mp I_0^1)(I^1 \pm I_0^1 + 1)} \hat{D}_{I^1 I_0^1 \pm 1}^* , \\ [\hat{I}_0, \hat{D}_{I^1 I_0^1}^*] &= I_0^1 \hat{D}_{I^1 I_0^1}^* . \end{aligned} \quad (4.3)$$

Then, we define the following state:

$$|I^1 I^0, II_0; T\rangle = \sum_{I_0^1 I_0^0} \langle I^1 I_0^1 I^0 I_0^0 | II_0 \rangle \hat{D}_{I^1 I_0^1}^* |I^0 I_0^0; T\rangle . \quad (4.4)$$

Here, $\langle I^1 I_0^1 I^0 I_0^0 | II_0 \rangle$ denotes the Clebsch-Gordan coefficient. Clearly, the state $|I^1 I^0, II_0; T\rangle$ is an eigenstate of $\tilde{\mathbf{T}}^2$, \tilde{T}_0 , \tilde{R}_0 , $\hat{\mathbf{I}}^2$, \hat{I}_0 and \hat{M}_0 :

$$\text{the eigenvalue of } \begin{cases} \tilde{\mathbf{T}}^2 = T(T-1) , \\ \tilde{T}_0 = T , \\ \tilde{R}_0 = R , \quad (R = 2I^0 - (T - 3/2)) \\ \hat{\mathbf{I}}^2 = I(I+1) , \\ \hat{I}_0 = I_0 , \\ \hat{M}_0 = 3I^1 - M^0 . \quad (M^0 = 2(T - 3/2) - I^0) \end{cases} \quad (4.5)$$

Further, we have

$$\tilde{T}_- |I^1 I^0, II_0; T\rangle = 0 . \quad (4.6)$$

For the proof of the last relation in Eq.(4.5), we used

$$[\hat{M}_0, \hat{D}_{I^1 I_0^1}^*] = 3I^1 \hat{D}_{I^1 I_0^1}^* . \quad (4.7)$$

Of course, the state $|I^1 I^0, II_0; T\rangle$ is the eigenstate of $\hat{I}_{su(3)}$ and (I^1, I^0, I) obeys

$$|I^1 - I^0| \leq I \leq I^1 + I^0 . \quad (4.8)$$

If the discussion is restricted only to the $su(3)$ -algebra, it is enough to take into account the set $\{|I^1 I^0, II_0; T\rangle\}$. If we include the $su(1, 1)$ -algebra, the state $|I^1 I^0, II_0; T\rangle$ is generalized

to the eigenstate of \tilde{T}_0 , the eigenvalue of which is different from T . The relation (4.6) leads us to

$$|I^1 I^0, II_0; TT^0\rangle = \sqrt{\frac{(2T-3)!}{(T_0-T)!(T_0+T-3)!}} (\tilde{T}_+)^{T_0-T} |I^1 I^0, II_0; T\rangle ,$$

$$T_0 = T, T+1, T+2, \dots \quad (T_0 \geq T) \quad (4.9)$$

Following an idea developed in the above, we have the orthogonal set for the $su(3)$ -algebra and its associating $su(1,1)$ -algebra.

§5. An orthogonal set apparently different from the form in §4

As is clear from the treatment presented in §4, the derivation of the eigenstate (4.9) is based on the fact that (\hat{D}_\pm^*) forms a spherical tensor with rank 1/2. In this section, we construct an orthogonal set in the framework of a set of two excited state generating operators which does not form the spherical tensor. It may be characteristic that, in contrast to the form obtained in §4, the form presented in this section is monomial.

First, we introduce the following operators:

$$\mathbf{D}_-^* = \hat{D}_-^* , \quad \mathbf{D}_+^* = \hat{D}_+^* \cdot (\hat{I}_0 - 1/2) + (1/2) \hat{I}_+ \cdot \hat{D}_-^* . \quad (5.1)$$

They obey the relations

$$[\hat{I}_- , \mathbf{D}_-^*] = 0 , \quad [\hat{I}_0 , \mathbf{D}_-^*] = -(1/2) \mathbf{D}_-^* , \quad (5.2a)$$

$$[\hat{I}_- , \mathbf{D}_+^*] = \mathbf{D}_+^* \cdot \hat{I}_- , \quad [\hat{I}_0 , \mathbf{D}_+^*] = (1/2) \mathbf{D}_+^* , \quad (5.2b)$$

$$[\hat{M}_0 , \mathbf{D}_\pm^*] = (3/2) \mathbf{D}_\pm^* , \quad (5.3)$$

$$[\mathbf{D}_+^* , \mathbf{D}_-^*] = 0 , \quad (5.4)$$

$$\mathbf{D}_- |I^0, T\rangle = 0 , \quad \mathbf{D}_+ |I^0, T\rangle = 0 . \quad (5.5)$$

With the use of the operators \mathbf{D}_\pm^* , we introduce the following state:

$$\|I^1 I^0, I; T\rangle = (\mathbf{D}_-^*)^{I^1-I^0+I} (\mathbf{D}_+^*)^{I^1+I^0-I} |I^0, T\rangle . \quad (5.6)$$

Here, the normalization constant is omitted. The state (5.6) obeys the following relations:

$$\tilde{T}_- \|I^1 I^0, I; T\rangle = 0 , \quad (5.7a)$$

$$\tilde{T}_0 \|I^1 I^0, I; T\rangle = T \|I^1 I^0, I; T\rangle , \quad (5.7b)$$

$$\tilde{R}_0 \|I^1 I^0, I; T\rangle = R \|I^1 I^0, I; T\rangle , \quad (5.7c)$$

$$\hat{I}_- \|I^1 I^0, I; T\rangle = 0 , \quad (5.8a)$$

$$\hat{I}_0 \|I^1 I^0, I; T\rangle = -I \|I^1 I^0, I; T\rangle , \quad (5.8b)$$

$$\hat{M}_0 \|I^1 I^0, I; T\rangle = (3I^1 - M^0) \|I^1 I^0, I; T\rangle . \quad (5.8c)$$

Here, R and M^0 are given in the relations (4.5).

The relations (5.7) and (5.8) permit us to introduce the state $\|I^1 I^0, II_0; TT_0\rangle$ in the form

$$\|I^1 I^0, II_0; TT_0\rangle = (\tilde{T}_+)^{T_0-T} (\hat{I}_+)^{I+I_0} \|I^1 I^0, I; T\rangle . \quad (5.9)$$

The state (5.9) is the eigenstate of $\tilde{\mathbf{T}}^2$, \tilde{T}_0 , \tilde{R}_0 , $\hat{\mathbf{I}}^2$, \hat{I}_0 and \hat{M}_0 with the eigenvalues $T(T-1)$, T_0 , R , $I(I+1)$, I_0 and $3I^1 - M^0$, respectively. The exponents of \mathbf{D}_\pm^* appearing in the state (5.6) should be positive integers:

$$I^1 - I^0 + I \geq 0 , \quad I^1 + I^0 - I \geq 0 . \quad (5.10)$$

Further, we note the relation

$$\begin{aligned} \hat{I}_- (\mathbf{D}_+^*)^{I^1+I^0-I} |I^0, T\rangle &= 0 , \\ \hat{I}_0 (\mathbf{D}_+^*)^{I^1+I^0-I} |I^0, T\rangle &= [(1/2)(I^1 + I^0 - I) - I^0] (\mathbf{D}_+^*)^{I^1+I^0-I} |I^0, T\rangle . \end{aligned} \quad (5.11)$$

Since $(1/2)(I^1 + I^0 - I) - I^0 = (1/2)(I^1 - I^0 - I)$, the state $(\mathbf{D}_+^*)^{I^1+I^0-I} |I^0, T\rangle$ can be regarded as the state $|(1/2)(I + I^0 - I^1), T\rangle$. Therefore, $(1/2)(I + I^0 - I^1)$ should be positive integer:

$$I + I^0 - I^1 \geq 0 . \quad (5.12)$$

Combining the relations (5.10) and (5.12), we have

$$|I^1 - I^0| \leq I \leq I^1 + I^0 . \quad (5.13)$$

Thus, we can learn that the state $\|I^1 I^0, II_0; TT_0\rangle$ is identical to the state $|I^1 I^0, II_0; TT_0\rangle$ defined in the relation (4.9). Of course, the normalization constant is omitted. The state (5.9) is monomial, i.e., the state (5.9) is obtained by successive operation of \tilde{T}_+ , \hat{I}_+ , \mathbf{D}_-^* and \mathbf{D}_+^* on the state $|I^0, T\rangle$.

§6. Discussion

In §§4 and 5, we presented the eigenvalue problem related to the $su(3)$ -algebra and its associating $su(1,1)$ -algebra. In this section, mainly, we discuss the case of the symmetric representation. The intrinsic state $|m\rangle$ in the present case can be expressed as

$$|m\rangle = |m_0, m_1\rangle = \left(\sqrt{m_0! m_1!} \right)^{-1} (\hat{b}_-^*)^{m_1} (\hat{b}_-^*)^{m_0} |0\rangle . \quad (6.1)$$

Here, m_0 and m_1 , which are used in (A), are given as

$$m_0 = T - 3/2 - R = 2[(T - 3/2) - I^0] , \quad m_1 = T - 3/2 + R = 2I^0 . \quad (6.2)$$

In (A), we showed that the cases $m_1 = 0$ and $m_0 = 0$ correspond to the symmetric representation, respectively. In this section, we discuss these cases in more detail. For this purpose, we note the relation

$$(\tilde{T}_0 - 3/2 - \tilde{R}_0)|I^1 I^0, II_0; T\rangle = m_0|I^1 I^0, II_0; T\rangle , \quad (6.3a)$$

$$(\tilde{T}_0 - 3/2 + \tilde{R}_0)|I^1 I^0, II_0; T\rangle = m_1|I^1 I^0, II_0; T\rangle , \quad (6.3b)$$

The above is derived from the relations (3.3) and (6.2). The relations (2.11), (2.14) and (6.3) lead us to

$$(\hat{b}^* \hat{b} + \hat{a}_+^* \hat{a}_+ + \hat{a}_-^* \hat{a}_-)|I^1 I^0, II_0; T\rangle = m_0|I^1 I^0, II_0; T\rangle , \quad (6.4a)$$

$$(\hat{a}^* \hat{a} + \hat{b}_+^* \hat{b}_+ + \hat{b}_-^* \hat{b}_-)|I^1 I^0, II_0; T\rangle = m_1|I^1 I^0, II_0; T\rangle , \quad (6.4b)$$

Let us start in the case $m_1 = 0$. In this case, the relation (6.2) tells us that $I^0 = 0$, and then, $I^1 = I$: $|I0, II_0; T\rangle$. Since $m_1 = 0$, the relation (6.4b) is equivalent to

$$\hat{a}^* \hat{a}|I0; II_0; T\rangle = \hat{b}_+^* \hat{b}_+|I0; II_0; T\rangle = \hat{b}_-^* \hat{b}_-|I0; II_0; T\rangle = 0 . \quad (6.5)$$

The relation (6.5) shows that the state $|I0, II_0; T\rangle$ does not contain \hat{a}^* , \hat{b}_+^* and \hat{b}_-^* . Therefore, if the treatment is restricted to the set $\{|I0, II_0; T\rangle\}$, which is a subspace of the space $\{|I^0 I^0, II_0; TT_0\rangle\}$, it is not necessary to take into account the bosons (\hat{a}, \hat{a}^*) , (\hat{b}_+, \hat{b}_+^*) and (\hat{b}_-, \hat{b}_-^*) , explicitly. Then, omitting these boson operators, the $su(3)$ -generators are expressed in the form

$$\hat{I}_+ = \hat{a}_+^* \hat{a}_- , \quad \hat{I}_- = \hat{a}_-^* \hat{a}_+ , \quad \hat{I}_0 = (1/2)(\hat{a}_+^* \hat{a}_+ - \hat{a}_-^* \hat{a}_-) , \quad (6.6a)$$

$$\hat{M}_0 = -\hat{b}^* \hat{b} + (1/2)(\hat{a}_+^* \hat{a}_+ + \hat{a}_-^* \hat{a}_-) , \quad (6.6b)$$

$$\hat{D}_-^* = \hat{a}_-^* \hat{b} , \quad \hat{D}_+^* = \hat{a}_+^* \hat{b} , \quad \hat{D}_- = \hat{b}^* \hat{a}_- , \quad \hat{D}_+ = \hat{b}^* \hat{a}_+ . \quad (6.6c)$$

The operators $(\tilde{T}_{\pm,0})$ and \tilde{R}_0 are reduced to

$$\tilde{T}_{\pm} = 0 , \quad (6.7a)$$

$$\tilde{T}_0 - 3/2 = -\tilde{R}_0 = (1/2)(\hat{b}^* \hat{b} + \hat{a}_+^* \hat{a}_+ + \hat{a}_-^* \hat{a}_-) . \quad (6.7b)$$

We can learn that the form (6.6) is identical with the symmetric representation.

Next, we consider the case $m_0 = 0$. As was discussed briefly in (A), this case essentially belongs to the symmetric representation. In order to show this fact explicitly, we relabel the boson operators as follows:

$$\begin{aligned} \hat{a} &\longrightarrow \hat{a}_+ , & \hat{a}_- &\longrightarrow \hat{b}_+ , & \hat{a}_+ &\longrightarrow \hat{a} , \\ \hat{b} &\longrightarrow \hat{b}_- , & \hat{b}_- &\longrightarrow \hat{b} , & \hat{b}_+ &\longrightarrow -\hat{a}_- . \end{aligned} \quad (6.8)$$

This procedure was sketched in (A) in different notations. Then, the $su(3)$ -generators shown in the form (2.10) are relabeled as

$$\hat{I}_+ \longrightarrow \hat{D}_-^* , \quad \hat{I}_- \longrightarrow \hat{D}_- , \quad \hat{I}_0 \longrightarrow (1/2)(\hat{M}_0 - \hat{I}_0) , \quad (6.9a)$$

$$\hat{M}_0 \longrightarrow (1/2)(\hat{M}_0 + 3\hat{I}_0) , \quad (6.9b)$$

$$\hat{D}_-^* \longrightarrow -\hat{I}_+ , \quad \hat{D}_+^* \longrightarrow \hat{D}_+^* , \quad \hat{D}_- \longrightarrow -\hat{I}_- , \quad \hat{D}_+ \longrightarrow \hat{D}_+ . \quad (6.9c)$$

The $su(1,1)$ -generators and \tilde{R}_0 are relabeled as

$$\tilde{T}_\pm \longrightarrow -\tilde{T}_\pm , \quad \tilde{T}_0 \longrightarrow \tilde{T}_0 , \quad (6.10)$$

$$\tilde{R}_0 \longrightarrow -\tilde{R}_0 . \quad (6.11)$$

Therefore, we can conclude that under the above relabeling, the case $m_0 = 0$ is reduced to the case $m_1 = 0$ which was already discussed.

In the next paper, on the basis of the relation (6.4), we will present the Holstein-Primakoff boson realization.

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Appendix A

—— Third form of the orthogonal set ——

In this Appendix, we show the third form of the orthogonal set. The first and the second forms are shown in §4 and §5, respectively. For the preparation, we consider the following operator:

$$\begin{aligned} \hat{P} &= 1 - \hat{I}_+(\hat{I}_-\hat{I}_+ + \epsilon)^{-1}\hat{I}_- , & \hat{I}_-\hat{I}_+ &= \hat{I}^2 - \hat{I}_0(\hat{I}_0 + 1) , \\ \epsilon &: \text{infinitesimal parameter} . \end{aligned} \quad (A.1)$$

Here, $(\hat{I}_{\pm,0})$ obeys the $su(2)$ -algebra, for example, shown in the relation (2.10a). In order to avoid null denominator, ϵ is introduced. Let $|I, I_0\rangle$ be the eigenstate of $\hat{\mathbf{I}}^2$ and \hat{I}_0 with the eigenvalues $I(I+1)$ and I_0 , respectively. Then, we have the following relation:

$$\begin{aligned} \hat{P}|I, I_0\rangle &= \epsilon((I+I_0)(I-I_0+1)+\epsilon)^{-1}|I, I_0\rangle \\ &\xrightarrow{\epsilon \rightarrow 0} \begin{cases} |I, -I\rangle, & (I_0 = -I) \\ 0. & (I_0 = -I+1, -I+2, \dots, I) \end{cases} \end{aligned} \quad (\text{A}\cdot 2)$$

Therefore, for an appropriately normalized eigenstate of \hat{I}_0 , $|I\rangle$, we have

$$\hat{P}|I\rangle = |I, -I\rangle. \quad (\text{A}\cdot 3)$$

We can see that the operator \hat{P} at the limit $\epsilon \rightarrow 0$ plays a role of the projection operator for pick up $|I, -I\rangle$ from $|I\rangle$.

Under the above preparation, we consider the $su(3)$ -algebra. First, we note the relations

$$[\tilde{T}_{\pm,0}, \hat{P}] = 0, \quad [\tilde{R}_0, \hat{P}] = 0, \quad (\text{A}\cdot 4)$$

$$\hat{I}_- \hat{P} = \epsilon \cdot (\hat{I}_- \hat{I}_+ + \epsilon)^{-1} \hat{I}_-, \quad (\text{A}\cdot 5\text{a})$$

$$\hat{P} \hat{I}_+ = \epsilon \cdot \hat{I}_+ (\hat{I}_- \hat{I}_+ + \epsilon)^{-1}, \quad (\text{A}\cdot 5\text{b})$$

$$[\hat{I}_0, \hat{P}] = 0, \quad (\text{A}\cdot 5\text{c})$$

$$[\hat{M}_0, \hat{P}] = 0. \quad (\text{A}\cdot 6)$$

Let us investigate the state $|I^1 I^0, I; T\rangle$ which is obtained by replacing \mathbf{D}_-^* and \mathbf{D}_+^* with \hat{D}_-^* and \hat{D}_+^* , respectively, in the state (5.6):

$$|I^1 I^0, I; T\rangle = (\hat{D}_-^*)^{I^1 - I^0 + I} (\hat{D}_+^*)^{I^1 + I^0 - I} |I^0, T\rangle. \quad (\text{A}\cdot 7)$$

The state (A.7) satisfies the same relations as those shown in the relations (5.7a)~(5.8c) except (5.8a):

$$\hat{I}_- |I^1 I^0, I; T\rangle \neq 0. \quad (\text{A}\cdot 8)$$

Then, we define the state

$$|I^1 I^0, I; T\rangle = \hat{P} |I^1 I^0, I; T\rangle. \quad (\text{A}\cdot 9)$$

The relations (A.3) and (A.5a) give us

$$\hat{I}_- |I^1 I^0, I; T\rangle \xrightarrow{\epsilon \rightarrow 0} 0. \quad (\text{A}\cdot 10)$$

With the use of the relations (A·4)~(A·6), we can prove the relations (5·7a)~(5·8c). Of course, $\{\|I^1I^0, I; T)\}$ is orthogonal. Then, in the same form as that shown in the relation (5·9), we have

$$\|I^1I^0, II_0; TT_0) = (\tilde{T}_+)^{T_0-T}(\hat{I}_+)^{I+I_0}\|I^1I^0, I; T) . \quad (\text{A}\cdot 11)$$

We can derive the state (A·11) by successive operation of \tilde{T}_+ , \hat{I}_+ , \hat{P} , \hat{D}_-^* and \hat{D}_+^* . The above is the third form for the orthogonal set of the $su(3)$ -algebra.

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